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New Time-Series Statistic for Detecting Rhythmic Co-Occurrence in the Frequency Domain: The Weighted Coherence and Its Application to Psychophysiological Research

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The relationships among physiological systems and between physiological and behavioral systems have frequently been evaluated with traditional descriptive statistics. These methods may not be sensitive to underlying rhythmic relationships. Although cross-spectral analysis provides a method for assessing rhythmic co-occurrence, it is only capable of describing the covariation of two systems at specific frequencies. Since physiological and behavioral systems tend not to be manifested at a constant frequency, a new statistic is described that is capable of evaluating the shared rhythmicity of two systems across an entire band of frequencies.

Psychophysiological research often describes parallels between physiological and behavioral response systems. Psychophysiological constructs such as *response fractionation* (Lacey, 1967) and *cardiac-somatic coupling* (Obrist, Webb, Sutterer, & Howard, 1970) have been defined in terms of the statistical relationship among physiological response systems. Even research evaluating the ubiquitous construct of arousal has been dependent on a statistical assessment of the relationship among response systems. Research assessing the validity of a general arousal theory has usually consisted of describing the central tendency (e.g., mean) or variability (e.g., range, variance) of specific physiological response systems and then correlating these variables. Data resulting in high interresponse system correlations have been used to support a general arousal or activation theory, whereas low interresponse system correlations have been used to critique the construct of arousal. However, it is possible that the descriptive statistics of mean and variance are not sensitive to all the underlying organizational characteristics of the nervous system. Specifically, mean and variance statistics are insensitive to rhythmicity, and the correlations among the descriptive statistics are insensitive to rhythmic co-occurrence. Thus the conclusions of earlier landmark studies may have been a function of the statistical methodology rather than of the underlying principles of neural and behavioral organization.

This article describes a new time-series statistic that may be used to assess the organization of physiological and behavioral response systems. Potentially this technique may be used to redefine constructs such as arousal. This article describes the modification of a class of time-series statistics: cross-spectral analysis. This modified statistic may be the appropriate method to describe the shared rhythmic variation between two physiological response systems. The relationship between heart rate and respiration will be used as a model system to

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illustrate the potential usefulness of the technology. These methods, however, may be applied to other behavioral or physiological response systems.

Time Series: A Definition

Although most psychophysiological data are presented in terms of mean levels within or across subjects, the sequential pattern of a physiological response for a specific subject during a period of time may provide important information. Time-series statistics provide methods to describe these patterns. A set of sequential observations, such as the amplitude of respiration sampled every second or the time intervals between sequential heart beats, constitutes a time series. Mathematically, a time series may be described as a string of random variables that are sequentially indexed, for example,

$$X_t,\ X_{t+1},\ X_{t+2}, \ldots,\ X_{t+n}.$$

In this example, the index t represents time. There are two basic approaches that may be used to describe and analyze a time series. The series may be represented and analyzed in the time domain or in the frequency domain. Time domain representations plot data as a function of time. Time domain methods of analysis are based on autocorrelation and cross-correlation measures. As their names imply, the techniques are mathematical extensions of traditional correlational techniques. An autocorrelation is the correlation of one time series with a time-shifted version of itself. If the time series is periodic, the plot of the autocorrelations (the autocorrelogram) at different time lags will be periodic. Similarly, a cross-correlation is the correlation of one time series with a time-shifted version of a second time series. The cross-correlation function provides information regarding the statistical dependence of one wave form with another. If the two time series are identical, the peak value of the cross-correlation function will be unity at the lag that makes the two series identical and less than unity at all other lags. In most cases, since the second series is not solely a time-shifted version of first series, the peak value of the cross-correlation will be less than unity.

Autocorrelation techniques are very effective

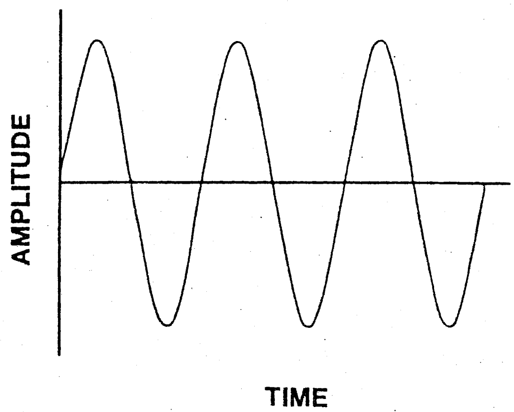

Figure 1. Pure sine wave.

in detecting periodicities only when the series are characterized by a relatively pure sinusoid, uncontaminated by other random influences. Cross-correlation techniques lose their effectiveness and sensitivity to assess the commonality between two series when the difference between series is more than a temporal displacement. Anyone can recognize the single sine wave in Figure 1; however, to recognize the components of a mixture of sine waves in Figure 2 may be difficult. In Figure 3, the four sine waves that were summed in Figure 2 are superimposed.

If the variance associated with the signal of interest represents only a small percentage of the total variance of the series, then the successful application of time domain techniques will be limited to the experimenter's ability to filter the data by removing periodicities other than the one of interest. This requires a priori knowledge of the underlying periodic structure of the process, the basic reason for performing the analysis in the first place and hence a priori unknown to the experimenter.

Frequency domain techniques are those based on the *spectral density function* that describes how the periodic variation in a time series may be accounted for by cyclic components at different frequencies. The procedure for estimating the spectral densities at various frequencies is called *spectral analysis*. For bivariate series, the cross-spectrum measures the covariances between two series at different frequencies. Spectral technology decomposes the variance of a time series into constituent frequencies or periodicities. There is a mathe-

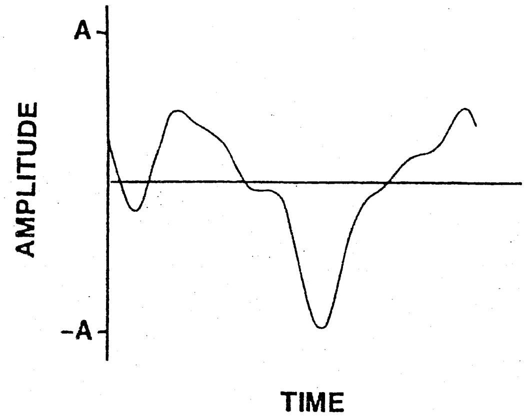

Figure 2. Sum of four pure sine waves.

matical relationship between the time domain correlation procedures and spectral analysis. The spectral density function is the Fourier transform of the autocovariance (unstandardized autocorrelation) function, and the cross-spectral density function is the Fourier transform of the cross-covariance function. The selection of time domain or frequency domain analysis for the study of rhythmicity is a function of the characteristics of the data set. In the study of many physiological processes and response systems, frequency domain analysis appears to be appropriate, since these systems are not manifested in pure sinusoids.

Periodic Covariation: A Physiological Example

A component of heart rate variability appears to covary with respiratory activity and is often referred to as respiratory-sinus arrhythmia (RSA). This covariation is obvious to the casual observer on inspection of the sequential measurements of both simultaneously measured variables, as illustrated in Figures 4 and 5.

Physiologically, RSA is a naturally occurring arrhythmia of the sinoatrial node that exhibits a periodicity (rhythm) similar to respiration. An increase in heart rate is observed during inspiration; heart rate decreases on expiration.

Although the mechanisms underlying the complex interaction resulting in RSA are unknown and are the focus of ongoing cardiopulmonary research, the mechanisms of specific components of RSA are understood. By carefully selecting the better understood components, cross-spectral analysis may be used to accurately describe the respiratory/heart rate interaction.

During inspiration, the lungs expand and stimulate stretch receptors that convey information to the brain stem. The stretch receptors are prepotent in signaling the brain stem to inhibit (gate) the vagal efferents to the heart that cause an increase in heart rate (see Lopes & Palmer, 1976). During exhalation, the stretch receptors in the lungs no longer convey information to the brain stem gating mechanism, and the vagal efferents influence the heart by prolonging the time between successive heart beats. Although heart rate activity is not solely determined by the respiratory influence on the vagus, time-series analyses may be conducted to assess the covariation of heart rate patterns and lung movements.

How can one assess the covariation of the heart rate and respiratory activity, given that there is a physiological basis for such an interaction? Research investigating this relationship has often been dependent on traditional descriptive statistics that have attempted to relate heart rate variability to respiration rate or amplitude. The research has been interpreted to support some level of interaction by identifying a covariation among decreased respiratory amplitude, increased respiratory frequency, and decreased heart rate variability (e.g., Cheung & Porges, 1977; Coles, 1972; Lacey, 1967; Porges & Raskin, 1969). These

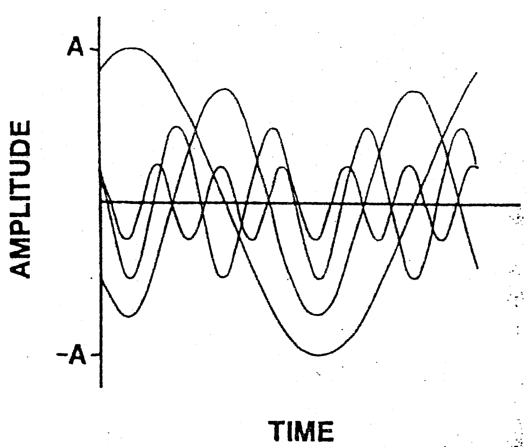

Figure 3. The constituent periodicities of Figure 2.

analyses, however, are not sensitive to the degree of periodic or temporal covariation between the two systems; the data only report that the systems respond in parallel when summed across subjects.

Spectral Analysis: A Method for Describing Periodic Processes

Spectral analysis is the natural tool for answering questions regarding periodic or rhythmic processes. Respiration and heart rate are inherently rhythmic. For example, the physiological construct of a central respiratory drive is periodic and is often described in terms of specific frequencies. Spectral analysis is merely a method of quantifying the periodicity and partitioning the total variance of the series into components associated with specific rhythms.

Although respiratory frequencies may be easily identified by observing a polygraph record, the frequency components of heart rate activity are more difficult to identify. The heart is driven and controlled by various neural, mechanical, and biochemical factors. Hypothetically, each factor may be associated with a specific periodicity.

Compare the sinusoidal characteristics of the temporal changes in respiration amplitude plotted in Figure 5 with the less periodic plot of the time intervals between sequential heart beats in Figure 4. These figures are time domain representations of the two physiological

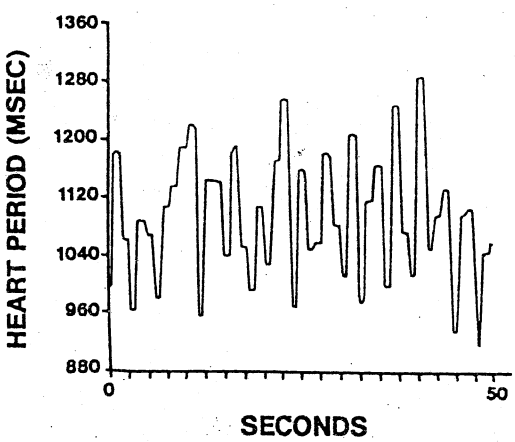

Figure 4. Second-by-second heart period values (time intervals between sequential heartbeats).

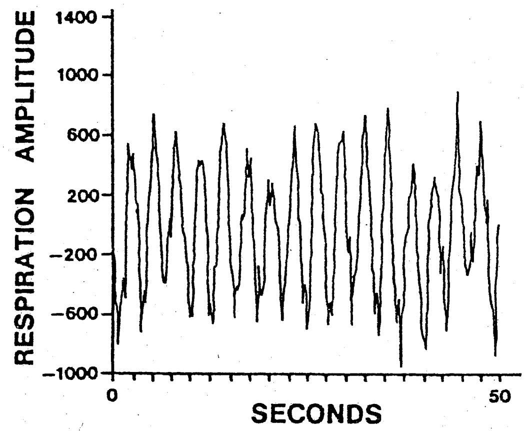

Figure 5. Second-by-second measures of respiration amplitude (changes in chest circumference).

processes, since the data are plotted as a function of time. Time domain representations are often the sum of the various sinusoids. As a general rule, with the exception of processes that are manifested by relatively pure sinusoids (e.g., respiration), it is difficult or impossible to identify rhythms by viewing a time series in the time domain. By viewing the time series of sequential heart periods in the frequency domain, spectral analysis decomposes the summed sinusoids into constituent frequencies.

The comparison of spectral analysis with analysis of variance may be helpful in facilitating an understanding of spectral methodology. Spectral analysis partitions the variance of a time series in a way that is strictly analogous to the partitioning of variance in the analysis of variance. The spectral density function decomposes the total mean squares (σ^2) into mean squares over the constituent frequency bands, just as the analysis of variance decomposes the total mean squares into orthogonal constituent mean squares, such as main effects, interactions, and errors.

Spectral analysis may be used to study rhythmic activity of heart rate and respiration by decomposing both series of sequential observations into constituent sinusoidal functions of different frequencies. The frequencies of interest in the study of RSA are the frequencies associated with the normal spontaneous respiratory activity. If the breathing were con-

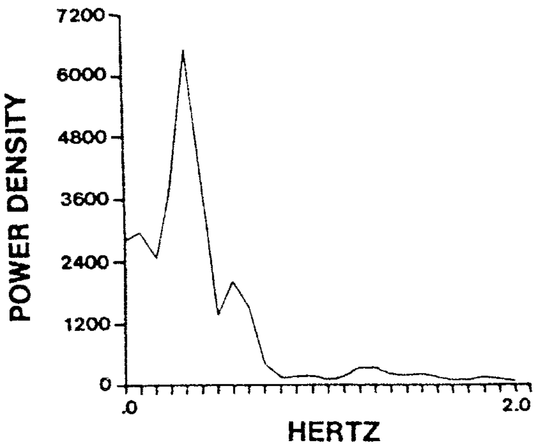

Figure 6. Spectral density function of the heart period data graphed in Figure 4.

stant at a rate of 12 times a min, each breath would take approximately 5 sec and would have a frequency of .2 cycles/sec (.2 Hz), since one fifth of the breath would occur in each sec.

Although it is necessary to introduce spectral concepts, this article does not discuss how the spectral density function is estimated, problems of spectral window design, aliasing, filtering, significance tests, or other related frequency domain concepts. (For a discussion of these points, see Bohrer & Porges's, in press, condensed overview, or Brillinger's, 1975, in-depth mathematical treatment.) The spectral density function is calculated to evaluate the variances of the specific frequencies of the decomposed heart rate and respiratory patterns by yielding power density scores at each different frequency. The power density is an estimate of the variance attributed to a specific frequency. In Figures 6 and 7, the power densities for each frequency of heart rate and respiration are plotted. These analyses were conducted on the data plotted in Figures 4 and 5. Note that both series exhibit a peak power density estimate at the same frequency, representing the dominant or most characteristic respiratory frequency.

To calculate the component of heart rate variability associated with respiratory activity, the spectral densities for each frequency characteristic of respiratory activity are accumulated. This is mathematically justified, since rhythms at different frequencies are known to be uncorrelated (Brillinger, 1975). In adults, the respiratory frequency band would represent rates between approximately 8 and 25 breaths/min; in children, it would be between 15 and 30 breaths/min.

The spectral density of the heart rate (H) process at a specific frequency may be represented as $f_H(\theta)$, whereas the spectral density of the respiratory (R) process at a specific frequency may be represented as $f_R(\theta)$. Spectral analysis may provide methods to quantify concepts such as *respiratory stability*. Respiratory stability could be assessed in terms of the percentage of total variance of the respiration series associated with a specific band of frequencies. Even though the spectral decomposition of both respiration and heart rate processes results in similar dominant frequencies, the heart rate process exhibits other prominent frequencies, independent of respiration, that have been theoretically associated with physiological processes such as temperature and blood pressure fluctuations (Chess, Tam, & Calaresu, 1975).

Cross-Spectral Analysis: A Method for Describing Periodic Covariation

Cross-spectral analysis generates a coherence function, a measure of the best linear association of each observed rhythm in one variable on the same rhythm in the second variable. The coherence $[\rho^2(\theta)]$ as described in the following equation is the square of the correla-

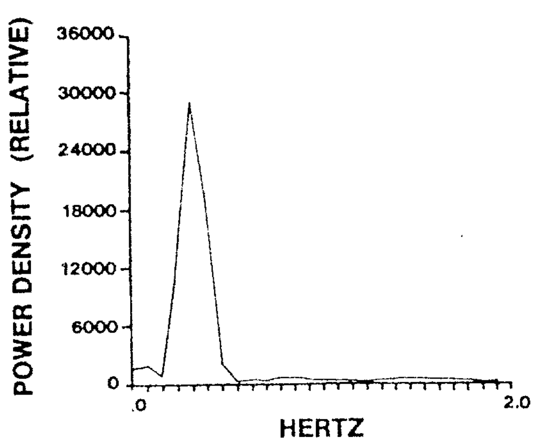

Figure 7. Spectral density function of the respiration data graphed in Figure 5.

tion between the components (sinusoids) of the two processes at a specific frequency (θ).

$$\rho^2(\theta) = \frac{|f_{HR}(\theta)|^2}{f_H(\theta)\, f_R(\theta)}.$$

The coherence at frequency θ is the square of the cross-spectral density $f_{HR}(\theta)$ divided by the product of spectral densities of each series at frequency θ. Note the similarity of this equation with the calculation of a squared correlation coefficient; the cross-spectral density parallels the squared cross-products and the spectral densities parallel the variances. Conceptually, the coherence may be thought of as a time-series analogue of the omega squared (Hays, 1963) or as the proportion of variance accounted for by the influence of one series on the other at each specific frequency. Figure 8 is a plot of the coherence spectrum of heart rate and respiration.

Since respiratory rhythms occur over a band of frequencies, the calculation of a summary statistic that describes the proportion of shared variance of the two systems is complex. The coherence function that describes the correlation at each frequency must be modified to summarize the covariation of heart rate and respiration across the respiratory frequency band. This modified coherence would enable one number to describe the general relationship or coupling between respiratory and heart rate rhythmicity. We note, as illustrated in Figure 8, that the coherence is not constant across all frequencies and that in Figure 7 the heart period activity is not equally distributed in all frequencies. If the spectral densities were equally distributed, an unweighted mean coherence (for all frequencies) would accurately describe the relationship. For any other situation, it is necessary to calculate a weighted coherence, which provides an exact measurement of the proportion of variance of one series that is shared between the two processes. The following equation defines a new function, the weighted coherence, C_w,[1] which is the proportion of total variance of the H process that is shared with the R process within the frequency band from θ_1 to θ_2. In our example, θ_1 to θ_2 includes the frequencies most representative of respiratory rhythms in adults; θ_1 is equivalent to 8 cycles/minute (.125 Hz), and θ_2 is equivalent to 25 cycles/minute (.42 Hz).

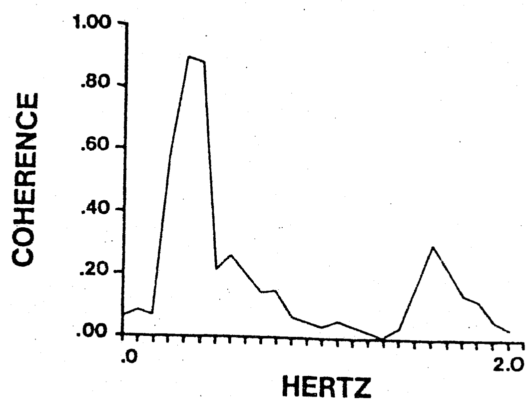

Figure 8. The coherence spectrum between the heart period data and respiration data graphed in Figures 4 and 5.

$$C_w = \frac{\int_{\theta_1}^{\theta_2} \rho^2(\theta)\, f_H(\theta)\, d\theta}{\int_{\theta_1}^{\theta_2} f_H(\theta)\, d\theta}.$$

It can be mathematically proven (see the Appendix) that the C_w, defined in the previous formula, is the only weighting of the coherences that provides the exact proportion of X explained by Y on the band θ_1 to θ_2. Thus the proportion of heart period activity shared with respiration within the dominant respiratory frequencies may be calculated. In our example, the estimate of C_w may be evaluated as described in the following equation:

$$C_w = \frac{\sum_{\theta_1}^{\theta_2} \rho^2(\theta)\, \hat{f}_H(\theta)}{\sum_{\theta_1}^{\theta_2} \hat{f}_H(\theta)}.$$

It is possible to have similar spectra for both processes during states in which the processes are totally unrelated. The relationship between the two spectral density functions and the coherence spectrum becomes clear if we describe hypothetical examples. Suppose an individual is exhibiting both rhythmic breathing and rhythmic heart period activity. When spectral analyses are performed on both series, the dom-

[1] The term *weighted coherence* is used by Galbraith (1966) to assess the coupling of cortical responses through various pairs of electrodes. The definition of Galbraith's weighted coherence $\bar{C}$ is different; instead of weighting the coherence by the spectral density estimate $\hat{f}_H$, it is weighted by the cross-amplitude density estimate. Galbraith's definition forfeits the interpretation of shared variance described previously, an important property of C_w.

inant frequency of both spectra are the same. This can be evaluated by using significance tests to assess whether a specific frequency is accounting for more variance than would be expected by chance (see Bohrer & Porges, in press). The identification that two series have the same dominant frequency provides no hint regarding the co-occurrence of the two processes and therefore no information regarding the coherence. The analogue of this situation in descriptive statistics would be the correlation between two series that have the same mean and variance, since knowledge of the mean and variance of two series does not provide information regarding their correlation.

Physiological Interpretation of the Cross-Spectral Analysis

The weighted coherence C_w may provide a quantitative estimate of stretch receptor influence on heart period activity. The brain stem may mediate a stretch receptor input/vagal output system along a continuum of efficiency from total co-occurrence (stretch receptor activity accounting for all heart period activity in the frequency band θ_1 to θ_2), $C_w = 1$, to total unrelatedness of the two processes, $C_w = 0$.

The coherence between the heart period process and the respiration process is dependent on processes occurring in the brain stem. For example, if the information from the stretch afferents is transformed in the brain stem by adding a random time delay to this information, it is possible that the output heart period spectrum may have characteristics similar to the input respiration spectrum (e.g., the same dominant frequency), although the coherence would be different than unity. However, if the afferent input were transformed by a constant time delay (uncontaminated by other random components), perhaps representing a constant neural transmission time, the coherence would be 1. Thus C_w could reflect the general coupling of these two processes and may manifest the degree of disruption within the brain stem.

Discussion

This article describes a new statistic that is capable of summarizing the proportion of shared variance of two processes that, although periodic, are manifested in an entire band of frequencies. In biological and behavioral systems, the variance of virtually all processes when they are spectrally analyzed will be decomposed over more than one frequency. In fact only a pure sinusoid, a rare process in any real-life system, will have its total variance associated with one frequency. In many biological systems the multiple determinants produce roughly periodic activity across a wide band of frequencies. The weighted coherence statistic C_w provides a mathematically justified procedure to summarize the proportion of shared variances between two systems over an entire band of frequencies. The application of C_w is not limited to the assessment of the coupling between respiration and heart period activity but may be used to assess the proportion of shared variance between any two processes that fit the statistical assumptions for spectral analysis. In preliminary work we have applied these techniques to assess the relationship between other pairs of processes: heart period and motor activity, heart period and blood pressure. Potentially these methods could be used to describe the covariation of two behavioral systems (see Gottman, 1979) or the covariation of a physiological and behavioral system. Thus the development of the C_w statistic provides the methodology to reevaluate psychophysiological constructs theoretically dependent on the co-occurrence of more than one response system.

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Appendix

Derivation of the Weighted Coherence

Here, the weighted coherence, as discussed in the text, is shown to be the natural representation of the proportion of variance of a time series $X(t)$ shared by the time series $Y(t)$ on the frequency band (θ', θ''). We find the best prediction of $X(t)$ on the band, as a function of $Y(t)$, and show that this predictor explains an amount of the $X(t)$ variance on the band (θ', θ'') that is equal to $2\int_{\theta'}^{\theta''} f_x(\theta)\, \rho^2(\theta)\, d\theta$ or explains a proportion of that variance equal to our definition of the weighted coherence, namely, $\int_{\theta'}^{\theta''} f_x(\theta)\, \rho^2(\theta)\, d\theta / \int_{\theta'}^{\theta''} f_x(\theta)\, d\theta$.

This derivation requires the spectral decomposition theorem, as derived briefly and eloquently in Hannan's (1960) Appendix. Denote the complex exponential function by $\exp(\cdot)$ and the conjugate of a complex number z by z^*. The spectral decomposition represents $X(t) = \int_{-\pi}^{\pi} \exp(i\theta t)\, dZ_x(\theta)$ and $Y(t) = \int_{-\pi}^{\pi} \exp(i\theta t)\, dZ_y(\theta)$, where Z_x and Z_y have orthogonal increments; that is, $E[dZ_x(\theta_1)\, dZ_x(\theta_2)^*] = E[dZ_y(\theta_1)\, dZ_y(\theta_2)^*] = E[dZ_x(\theta_1)\, dZ_y(\theta_2)^*] = 0$, if the θ_1 and θ_2 intervals $\pm[\theta_1, \theta_1 + d\theta_1]$ and $\pm[\theta_2, \theta_2 + d\theta_2]$ do not overlap, where $E|dZ_x(\theta_1)|^2 = \int_{\theta_1}^{\theta_1+d\theta_1} f_x(\theta)\, d\theta$ and $E|dZ_y(\theta_2)|^2 = \int_{\theta_2}^{\theta_2+d\theta_2} f_y(\theta) d\theta$, and where $Z(-\theta_1) = [-Z(\theta_1)]^*$ for $Z = Z_y$ or Z_x. Thus if f_{xy} denotes the Fourier transform of the cross-covariance function of X and Y, we have $\int_{\theta_1}^{\theta_1+d\theta_1} \exp(i\theta t) f_{xy}(\theta)\, d\theta = \int_{\theta_1}^{\theta_1+d\theta} \exp(i\theta t)\, E[dZ_x(\theta) dZ_y(\theta)^*]$, and $E[dZ_u(\theta)\, dZ_v(\theta)] = 0$ for $u, v = x$ or y. [Recall that the $Z(\theta)$ are complex random variables.]

We want to predict the part of $X(t)$ on the frequency band (θ', θ''), that is, to predict $X_K(t) = \int_{-\pi}^{\pi} K(\theta) \exp(i\theta t)\, dZ_x(\theta)$, where $K(\theta) = 1$ on $\pm(\theta', \theta'')$ and $K(\theta) = 0$ elsewhere. Consider predictors of the form $P = \int_{-\pi}^{\pi} P(\theta) \exp(i\theta t)\, dZ_y(\theta)$, and minimize, by choice of P, the "prediction variance," $E|X_K(t) - P|^2 = E|\int_{-\pi}^{\pi} \{K(\theta) \exp(i\theta t)[dZ_x(\theta) - f_{xy}(\theta)\, dZ_y(\theta)/f_y(\theta)]\} + \{[K(\theta)\, f_{xy}(\theta)/f_y(\theta) - P(\theta)]\, dZ_y(\theta)\}|^2$. Next, use the orthogonality of Z increments to write this as the mean integrated modulus square of the first term in { }, A, plus that of the second term in { }, B, plus the expected integrated cross-product of the terms in { }, C.

The spectral decomposition facts show that this expected squared integral is the double sum over those squares that are cross-products of Riemann-Stieltjes subintervals containing the major diagonal. We thus have $A = 2\int_{\theta'}^{\theta''} f_x(\theta)\, [1 - \rho^2(\theta)]\, d\theta$ and $C = 0$, whereas B is nonnegative. Since A does not involve P and B can be minimized by using the (optimum) predictor function $P(\theta) = K(\theta) f_{xy}(\theta)/f_y(\theta)$, the amount of X variance not explained by the best Y-based predictor P is exactly A. Thus the amount of X variance explained by Y on (θ', θ'') is $2\int_{\theta'}^{\theta''} f_x(\theta)[1 - \rho^2(\theta)]\, d\theta$, so that the proportion of the total X variance, $2\int_{\theta'}^{\theta''} f_x(\theta)\, d\theta$, is exactly the weighted coherence, as defined in this article.

Pierce (1979) defined a related but different weighted coherence.

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